Lecture 8

Sequences

<u>Def.</u> A sequence is a list of numbers written in order

$$\{a_n\} = \{a_1, a_2, a_3, \dots\}, \quad a_n \in \mathbb{R}.$$

The <u>n-th</u> term of the sequence is the n-th number on the list.

<u>Example 1.</u> In the sequence

$\{1, 2, 3, 4, 5, 6, \dots\}$ $a_1 = 1,\ a_2 = 2,\ a_3 = 3, \dots$

<u>Example 2</u> Some sequences can have patterns

$$\{1, -1, 1, -1, \dots\}$$

$a_{2j-1} = 1, \quad a_{2j} = -1 \quad , j = 1, 2, \dots$

Sequences can be represented in different ways

$$\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots \right\}$$

$$\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}, \qquad a_n = \frac{n}{n+1}.$$

Def. Limit of a Sequence (remind) recall.

A sequence $\{a_n\}$ has limit A if $\forall \varepsilon > 0 \quad \exists\, n_0 = n_0(\varepsilon)$ such that

$$|a_n - A| < \varepsilon \quad \forall n > n_0$$

$$\boxed{\lim_{n\to\infty} a_n = A}$$

If a finite limit A exists, we say that the sequence converges or is convergent. Otherwise, we say the sequence diverges

Definition

An infinite series is the sum of the terms of an infinite sequence

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

Example

$$\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots \right\}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

The sum of an infinite series

We define a new sequence of partial sums: $\{S_n\}$

$$S_1 = a_1, \quad S_2 = a_1 + a_2; \quad S_3 = a_1 + a_2 + a_3; \ldots$$

$$S_n = \sum_{j=1}^{n} a_j$$

Def. An infinite series $\sum_{n=1}^{\infty} a_n$ has a sum if the partial sums form a sequence $\{S_n\}$ that has a real limit

$$\lim_{n\to\infty} S_n = S.$$

We say ~~that infinite series~~ $\sum_{n=1}^{\infty} a_n$ converges. Otherwise, the series diverges (is divergent).

Example. Let us consider the following infinite series

$$a + aq + aq^2 + \dots + aq^n + \dots, \quad a \neq 0.$$

$$S_n = a\,\frac{1-q^n}{1-q}, \quad |q| \neq 1$$ (a geometric progression),

If $|q|<1$, then $q^n \to 0$, $n\to\infty$ and

$$\lim_{n\to\infty} S_n = \frac{a}{1-q}$$

The geometric progression series is converging.

If $|q|>1$, then $q^n \to \infty$, $n\to\infty$ and this series diverges.

Example 2. Find a sum of the ~~seque~~ series:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{n(n+1)}=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

First we calculate S_n:

$$S_n=\sum_{k=1}^{n}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=1-\frac{1}{n+1}$$

Thus

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1.$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1.}$$

Lemma 1. If S is a convergent series

$$S = \sum_{n=1}^{\infty} a_n$$

and we add a finite number k of new terms (positive or negative) then the obtained new series is also convergent.

Proof. Let's denote the new series

$$\tilde{S} = b_1 + b_2 + \dots + b_k + a_1 + a_2 + \dots$$
$$= b_1 + b_2 + \dots + b_k + S$$

$$\tilde{S} = \sum_{n=1}^{\infty} c_n ,$$

where $$c_n = \begin{cases} b_n, & 1 \le n \le k \\ a_{n-k}, & n > k. \end{cases}$$

Next we calculate a sequence of partial sums

$$\{\tilde{S}_n\}, \quad \tilde{S}_n = \sum_{k=1}^{n} c_n$$

If n is sufficently large; $(n > k)$

then

$$\tilde{S}_n = \underbrace{b_1 + \dots + b_k}_{B} + S_{n-k}$$

$$\lim_{n \to \infty} \tilde{S}_n = B + \lim_{n \to \infty} S_{n-k} = B + S$$

Thus the new series $\tilde{S}$ is convergent

Lemma 2. If $\sum_{n=1}^{\infty} a_n$ is a convergent series and

$$\sum_{n=1}^{\infty} a_n = S,$$

then if c is a constant we get that

$\sum_{n=1}^{\infty} (c a_n)$ is also convergent and

$$\sum_{n=1}^{\infty} (c a_n) = n S$$

Lemma 3. If two series $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ are convergent the a new series

$\sum_{n=1}^{\infty} (a_n + b_n)$ is also convergent

and

$$\sum_{n=1}^{\infty} (a_n + b_n) = A + B$$

Theorem 1. If series $\sum_{n=1}^{\infty} a_n$ converges then

$$\lim_{n\to\infty} a_n = 0.$$

Proof. Let's denote $S_n = \sum_{k=1}^{n} a_k$.

Since $\sum_{n=1}^{\infty} a_n$ converges then the following limit exists

$$S = \lim_{n\to\infty} S_n.$$

We can write as

$$S_n = S_{n-1} + a_n$$

and calculate limits

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n.$$

Since

$$\lim_{n\to\infty} S_{n-1} = S$$ it follows that

$$\lim_{n\to\infty} a_n = 0.$$

Example. Let's consider the following series

$$\sum_{n=1}^{\infty} \frac{n}{n+1}, \quad a_n = \frac{n}{n+1}$$

Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

thus the necessary convergence condition is not satisfied.

Remark. The condition $\lim_{n\to\infty} a_n = 0$ is only necessary, but not sufficient condition.

Example. The harmonic series is defined by

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

Proof. It is easy to show that (do it)

$$x > \ln(1+x), \quad \text{for } x > 0.$$

($f(x) = x - \ln(1+x)$ is a monotonic function

Let's consider partial sums

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

Since

$$1 > \ln 2, \quad \frac{1}{2} > \ln 3/2, \quad \frac{1}{3} > \ln\frac{4}{3}, \dots$$

then

$$S_n > \ln 2 + \ln(3/2) + \ln\left(\frac{4}{3}\right) + \dots \ln\left(\frac{n+1}{n}\right)$$

$$= \ln\left(2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \dots \frac{n+1}{n}\right) = \ln(n+1)$$

But we have that

$$\lim_{n\to\infty} (n+1) = \infty$$

thus S_n diverges. $S_n \to \infty, n \to \infty$

Cauchy's convergence test

The first test follows from results on testing the convergence of sequences

Th. 1 (recall the test).

A sequence of numbers $\{x_n\}$ converges iff (if and only if)

$\forall \varepsilon > 0$ there $\exists N : n > N(\varepsilon)$, $\forall p \in \mathbb{N}$

$\Rightarrow |x_{n+p} - x_n| < \varepsilon$

Let's consider a sequence of partial sums $\{S_n\}$

$$S_n = \sum_{k=1}^{n} a_k$$

$\forall \varepsilon > 0$ there $\exists N = N(\varepsilon)$: $n > N(\varepsilon)$

$\forall p \in \mathbb{N}$

$$|S_{n+p} - S_n| < \varepsilon$$

$$|a_{n+1} + a_{n+2} + \dots + a_{n+p}| < \varepsilon.$$

Example 1

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad a_n = \frac{1}{n^2}.$$

The necessary convergence condition is satisfied since:

$$\lim_{n \to \infty} \frac{1}{n^2} = 0.$$

We have the following estimates

$$\frac{1}{(n+1)^2} < \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\frac{1}{(n+2)^2} < \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$\cdots$$

$$\frac{1}{(n+p)^2} < \frac{1}{n+p-1} - \frac{1}{n+p}$$

and

$$\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+p)^2} < \frac{1}{n} - \frac{1}{n+p}.$$

$$< \frac{1}{n}.$$

By taking

$\forall \varepsilon > 0 \quad N = N(\varepsilon) = \left\lceil \frac{1}{\varepsilon} \right\rceil : n > N(\varepsilon)$

$\Rightarrow \boxed{\frac{1}{n} < \varepsilon.}$ $\Rightarrow$ The series converges

Example 2. Let's calculate

$$\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}, \quad a_n = \frac{1}{\ln(n+1)}$$

Again, the necessary convergence condition is satisfied

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0.$$

$$a_{n+1} + a_{n+2} + \dots + a_{n+p} > \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p}$$

$$> \frac{1}{n+p} + \frac{1}{n+p} + \dots + \frac{1}{n+p} = \frac{p}{n+p} = 1 - \frac{n}{n+p}.$$

If $p = n$ then

$$1 - \frac{n}{n+p} = 1 - \frac{1}{2} = \frac{1}{2}, \quad \text{thus}$$

$$a_{n+1} + a_{n+2} + \dots + a_{n+p} > \frac{1}{2}$$ (The series is divergent)